A New Monte-Carlo Simulation Model for the Temporal Development of Cloud Droplet Spectra

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ABSTRACT

A new Monte-Carlo simulation model for computing the temporal development of cloud droplet spectra is presented. Since the simulation is carried out in probability space, the new model does not need to follow the growth history and position of each and every drop. Additional physical processes, such as turbulence, electrical effects, glaciation, etc., can be added to the primary growth process of gravitational coagulation with relative ease. The main features of the model are demonstrated by several numerical examples.

1. Introduction

In an earlier paper Kornfeld *et al.* (1968) proposed a method for direct numerical simulation of the process of cloud droplet growth by accretion. However, the simplicity of the method, which raised hopes for a possible future inclusion of other complex physical processes (such as turbulence and electrical effects), was offset by the excessive demands this method presented in terms of computer memory and speed.

The basic reason for this was the fact that the simulation was carried out in geometrical space, individually following the growth history of each and every drop. In order to overcome this handicap, the present simulation model was devised. Here, geometrical space was dropped in favor of probability space, and groups of common-size drops were substituted for the individual drops in the old simulation model.

The new simulation model is described in Section 2. Section 3 shows some numerical examples, where some applications of the new method are demonstrated. Conclusions are drawn in Section 4.

2. Direct simulation in probability space

Assuming that all drops are randomly distributed throughout the cloud, with n_0 drops per unit volume per radius interval of radii $r_0 \leqslant r \leqslant r_1$, and n_i drops for $r_i \leqslant r \leqslant r_{i+1}$, then any group n_i is randomly distributed in any partial volume of the whole cloud. If the total volume under consideration is 1, then the probability p of a specific particle being found in a region whose volume is V will be $V/1=p$. Let $q=1-p$. Then the probability that exactly k particles distributed uniformly at random are found in the volume V is

$$\binom{n_j}{k} p^k q^{(n_j-k)}.$$

Thus, the distribution function is the binomial distribution. In particular, the number of j drops with radius r_j in a given volume V can be determined by sampling from a random number generator with a binomial distribution and a mean n_jV. (This is, of course, also the *expected number*.)

We now define V to be the volume swept, during a time interval Δt, by the effective cross section of a single drop of group n_i [$=\pi R_i^2$, where R_i is the effective radius for collection of a single drop of group n_j, or $R_i=y_{i,j}r_i$, where $y_{i,j}$ is the linear collision efficiency of (r_i,r_j)] as shown in Fig. 1, i.e.,

$$V=h_i\pi R_i^2=\pi v_i y_{i,j}^2 r_i^2 \Delta t, \tag{1}$$

where v_i is the fall velocity relative to r_j, of an individual particle of the group n_i, and h_i is the effective vertical distance, with respect to j drops, traveled by this particle during time Δt.

The simulation process is then carried out in the following simple fashion:

1) Choose drops from an initial spectrum, for a given partial volume of the cloud $\bar{V}$. Define $N_i=\bar{V}n_i$.
2) Define a time interval Δt.
3) Find, with the aid of the random number generator, the number of drops of size j to be collected by

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one drop of size i. This number, multiplied by N_i is

$$N_{i,j}=\begin{cases}\text{total number of } j \text{ drops to be collected}\\ \text{during the time interval } \Delta t \text{ by all } i \text{ drops}\end{cases}$$

4) Do (3) for all $i>j$, resulting in

$$\begin{array}{ll} N_{k,0} & N_{k-1,0}\cdots N_{i,0}\cdots N_{1,0}\\ N_{k,1} & N_{k-1,1}\cdots N_{i,1}\cdots N_{2,1}\\ \vdots & \vdots\\ N_{k,i} & N_{k-1,i}\cdots N_{i+1,i}\\ \vdots & \vdots\\ N_{k,k-2} & N_{k-1,k-2}\\ N_{k,k-1} & \end{array}$$

Column m gives the breakdown of the number of drops of sizes 0, 1, ..., $m-1$ to be collected during Δt by all m drops. Row n gives the breakdown of the number of drops of size n, to be collected during Δt by drops of sizes $n+1$, $n+2$, ..., k.

5) If

$$N_n<\sum_{l=n+1}^{k} N_{l,n}, \tag{2}$$

reset Δt to be half of its value and return to (3). Various other schemes were tried to optimize the choice of Δt. Initially, it was set to 10 sec. A typical scheme was to increase Δt by 10% if condition (2) were not satisfied and to multiply it by 0.5 otherwise. Such a scheme, it was hoped, would iterate around an optimal value. The problem, however, is that too many steps are then taken with a Δt that is too large, thus wasting computer time. Experimentation led to the conclusion that the halving scheme used for Δt was about as good as any, and had the advantages of simplicity.

6) Compute the spectrum at $t+\Delta t$ by

$$N_n(t+\Delta t)=N_n(t)-\sum_{l=n+1}^{k} N_{l,n}. \tag{3}$$

7) Compute the new radius of the group, $N_n(t+\Delta t)$, by

$$\left(r_n^3+\frac{1}{N_n}\sum_{j=0}^{n-1} N_{n,j}r_j^3\right)^{\frac{1}{3}}=r. \tag{4}$$

Eq. (3) shows the loss of drops in each category due to combinations with larger drops. Eq. (4) shows the growth of the *radii* of the larger drops. In this formulation, therefore, it is unnecessary to consider the formation of a large droplet from smaller droplets; all large droplets come from a process of accretion to a droplet treated as if it were large. This point may be understood more easily if it is realized that once the particles and associated radii are chosen from the appropriate distributions, each particle then has a fixed radius associated with it. This method preserves the liquid water content (LWC) of the cloud.

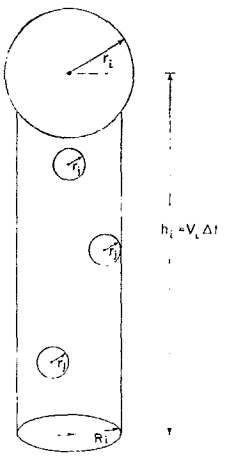

FIG. 1. Volume containing j drops to be swept by one i drop.

As each individual run, resulting from a repetitive application of sequences 1)–7), produces a particular realization, an "experiment" is defined as the temporal average of many such runs. It was found by experience that if "1000" is substituted for "many," a reasonably stable average is reached. Each of the 1000 initial distributions is formed by choosing a uniformly distributed random radius in each of 14 successive 2 μ intervals. The total expected LWC of that 2 μ interval is assigned to droplets of that chosen size.

3. Results

a. Temporal development of a cloud drop spectrum

Fig. 2 shows the temporal development resulting from the simulation model, of an initial cloud drop spectrum similar to the one used by Berry (1967), for geometric sweep-out. As explained above, this is an average of 1000 individual realizations, each using a different chain of random numbers. The function $g\langle \ln r\rangle$ is computed by using right-sided differences where differentials are called for except that at the first point in the graphs, $g\langle \ln r\rangle$ is approximated to be zero arbitrarily. The random number chain used was the standard one given in the IBM Scientific Subroutine Package.

In our approach, after the n_i drops are chosen at random from the specified distribution, a single radius between r_i and r_{i+1} is assigned to the whole group. This radius, renamed r_i, is allowed to change as these drops grow larger. It should be noted that the *number* of drops in each group becomes monotonically smaller as time progresses while the *size* becomes monotonically larger. Because of the vagaries of probablistic processes, it could happen that for $i<j$, $r_i>r_j$. In such a case, interchanges of notation would be made at the end of the step so that $r_i<r_j$ for $i<j$. Each point represents an average over 1000 trials.

As can be seen, a secondary peak which starts form-

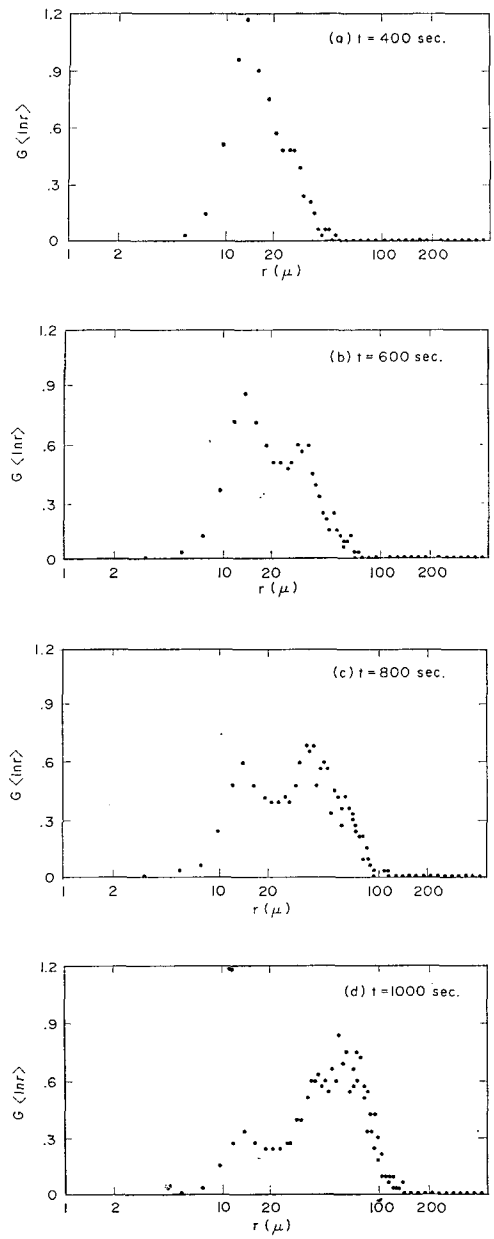

FIG. 2. Temporal development of a cloud droplet spectrum at 400, 600, 800 and 1000 sec for geometric sweep-out: $G\langle \ln r\rangle =$ constant $r^4 f(r)$, where f is the density function (cf. Berry, 1967).

ing at about $t=400$ sec, becomes quite pronounced at $t=600$ sec, and at $t=800$ sec becomes the dominant feature of the curve.

Compared to Berry's results, one notes that in the simulation model the liquid water starts to shift to drops of larger radii at a later time, while the primary peak (at $r\approx 11\ \mu$) persists longer.

A possible explanation of this discrepancy between the results of the two models may be found in Long's (1971) assumption [(a), p. 211)] that the time interval of the numerical integration should be small enough

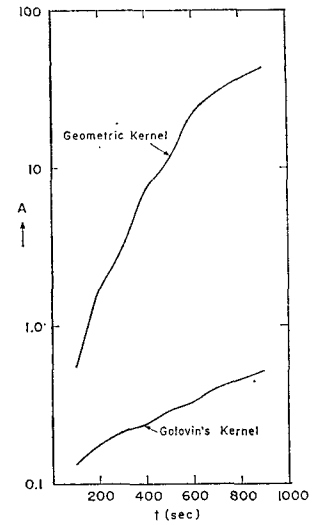

FIG. 3. Max $A(r_i,\Delta t)$ [defined in Eq. (5)] for $\Delta t=5$ sec.

so that only binary collisions occur during each Δt; the criterion for this condition to be fulfilled is

$$A(r_i,\Delta t)=\int_0^\infty pr(r_i,r_j;\,\Delta t)N(r_j,t)dr_j \ll 1, \qquad (5)$$

where $pr(r_i,r_j;\,\Delta t)$ is the probability that drop r_j will be collected by drop r_i during the time interval Δt, and $N(r_j,t)dr_j$ is the number of drops of radius between r_j and r_j+dr_j at time t.

Fig. 3 shows the maximum (over all i) of $A(r_i,\Delta t)$, as a function of time, for our reconstruction of Berry's integration of the coagulation equation. As can be seen, when Golovin's (1963) kernel is used, Long's condition is approximately satisfied, while when geometric sweep-out is assumed this condition is far from being met. In running several integrations, each with a smaller Δt, it became obvious that Δt would have to be made so small for Long's condition to be met as to make the integration impractical. These integrations were made with a replication of Berry's integration of the coagulation equation. The integration is an Euler method in time with the trapezoidal rule used in space. More complicated schemes gave the same or worse results.

b. *An experiment with a freezing mechanism in a supercooled cloud*

In order to demonstrate the simulation model's capacity of including various other physical processes along with the gravitational coagulation growth, an experiment was run in which the cloud was assumed to be supercooled, with freezing nuclei (number density of 1 cm^{-3}, radius 0.1 μ) interacting with the drops. When a drop captures an ice nucleus it is assigned a certain probability of freezing, and thus of becoming an ice particle. For a set of hypothetical conditions

[updraft of 2 m sec^{-1} and initial positioning of the simulated volume anywhere between the zero isotherm and the top of the cloud (assumed to be at −40C, which is also the temperature for homogeneous nucleation)], one gets for the above-mentioned probability

$$p \approx \frac{\Delta t}{2500}. \tag{6}$$

Ice particles can form by one of two alternatives: (i) capture by a water drop of an ice nucleus, and freezing at such time as determined by Eq. (6); and (ii) capture of a water drop by an ice particle, resulting in instantaneous freezing. Ice-ice coalescence was assumed not to occur.

Figs. 4 and 5 show the temporal development of the water drop and ice particle populations, respectively. As can be seen, very little happens up to about 600 sec, when glaciation starts. At $t=800$ sec, an appreciable part of the large drops (but virtually none of the smaller ones!) have frozen.

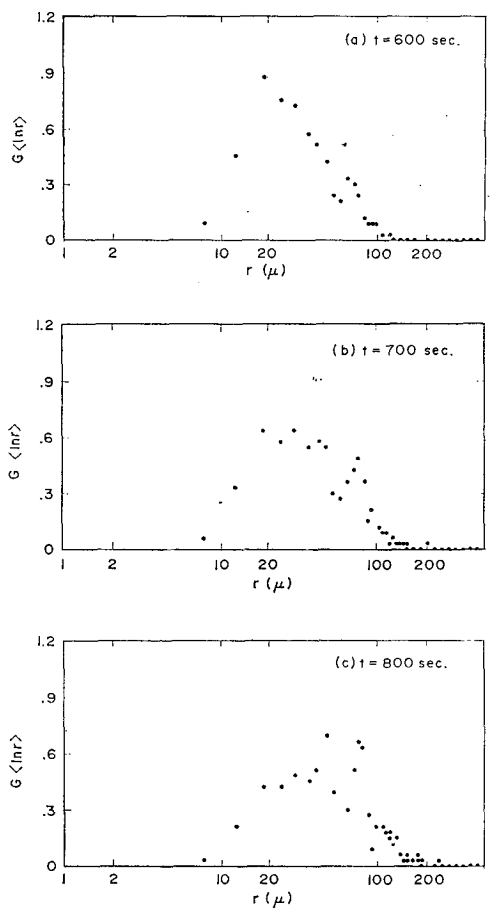

FIG. 4. Temporal development of a cloud droplet spectrum (water only), involving supercooled cloud with ice nucleation, at 600, 700 and 800 sec for geometric sweep-out: $G\langle \ln r\rangle =$ constant $r^4 f(r)$, where f is the density function (cf. Berry 1967).

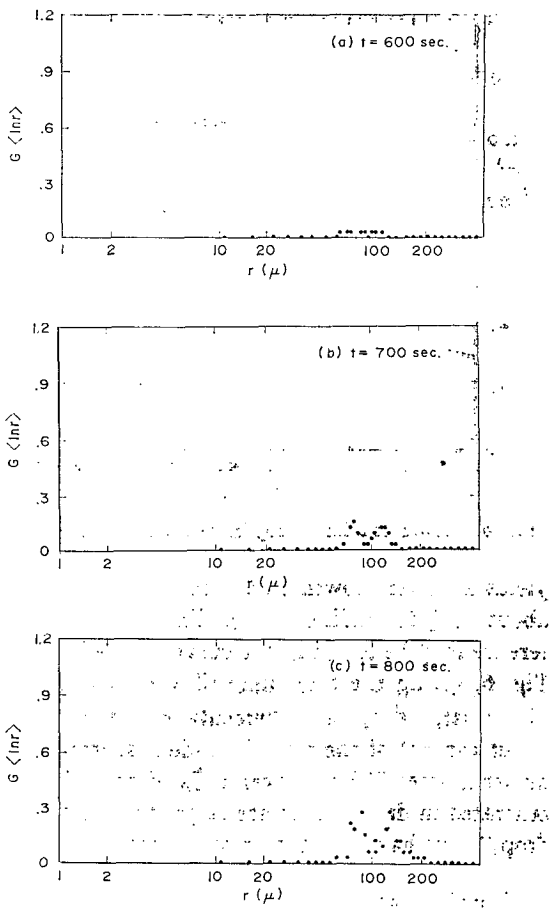

FIG. 5. As in Fig. 4, except for ice particles.

c. *The growth history of a drop falling through a changing spectrum*

Chin (1970) has carried out some Monte Carlo simulation experiments on the growth of a large drop in a polydispersed cloud, and compared his results with those predicted by the (continuous) growth equation. He concludes that spatial inhomogeneity contributes significantly to the broadening of the spectrum. While we concur with his general conclusions, we feel that the fact that he used a steady-state (rather than a continuously changing) spectrum may distort the actual effect.

Fig. 6 shows a comparison between two growth curves of a drop—one while falling through a steady-state spectrum (the same as Chin's), the other while falling through a spectrum which was initially identical to Chin's, but which was changed with time according to results obtained by integrating the coagulation equation. In both cases the droplets in the spectrum were treated by a randomization process as described in Section 3, at each time step. Thirty-two realizations were averaged in order to arrive at the curves shown in Fig. 6. As can be seen, the time-dependent experiment

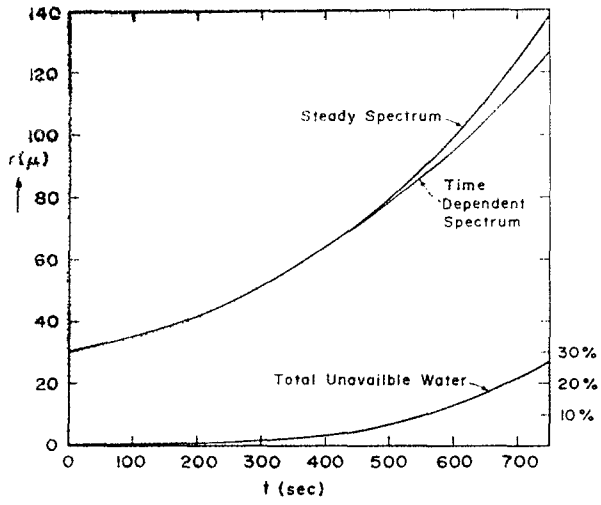

FIG. 6. Growth of a large drop in steady and changing spectra.

yields a slower growth curve; the difference gets to be about 10% for radii of 130 μ. The reason for this difference may be found in the curve on the lower part of Fig. 6, giving the total unavailable water for growth of the large drop, as a percentage of the total liquid water content of the time-dependent spectrum. As can be seen, after 750 sec, over 25% of the LWC is concentrated in drops which are larger than the "observed drop," thus hampering its growth rate.

4. Conclusions

A new, direct numerical simulation model was developed where cloud droplet growth by accretion is computed as a stochastic process. Based on our limited experimentation with this model, we conclude the following:

1) Compared to a previously suggested simulation model (in geometric space), the model is much more economical in terms of computer time.

2) This method is not subject to the limitation imposed by Long's criterion [Eq. (5)], unlike the numerical integration of the coagulation equation which fulfills it only in some special cases.

3) Additional physical processes (such as turbulence, sedimentation, electrical effects, nucleation, condensation, phase change, etc.) can be added to the primary process of gravitational coagulation with relative ease.

4) Statistical details of temporal spectrum development (such as the distribution of many individual realizations around the mean) would be readily available.

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